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Solutions to the 2 + 1 Toda equation

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Abstract. We use the inverse scattering transform method to obtain new real and bounded solutions of the 2 + 1 Toda equation. We present lump solutions depending on $3N$ complex parameters, solutions depending on arbitrary functions, periodic-like configurations which vanish as $t \to \infty$ and soliton and instanton configurations.

1. Introduction

In this paper we consider the 2 + 1 Toda equation:

$$(\partial_{xx} - \varepsilon^2 \partial_{tt})\theta(x, n, t) = 2\sigma^2\{e^{2(\theta_{n+1}-\theta_n)} - e^{2(\theta_n - \theta_{n-1})}\} \tag{1.1}$$

where $\theta(x, n, t) = \theta_n(x, t)$, $\sigma^2 = \pm 1$, $\varepsilon^2 = \pm 1$, x and t are continuous variables and n is a discrete coordinate. The above equation is natural to consider. In recent years it has been found that the classical soliton equations can be obtained as reductions of the self-dual Yang–Mills system and the Toda equation is one of the key reductions [1–3]. Moreover, x-independent solutions satisfy the Toda Lattice equation and suitable asymptotic reductions result in the well-known Kadomtsev–Petviashili and Davey–Stewartson systems [3]. Further it posseses the desirable property of being a system of coupled Lorentz invariant fields. Actually the 2 + 1 non-Abelian Toda system is a popular model in modern field theory and it also appears in general relativity. Finally we mention that it is integrable by means of the inverse scattering transform (IST, see [4, 5] for a review on 1 + 1 and 2 + 1 nonlinear integrable equations).

When $\sigma^2 = -\varepsilon^2 = -1$ the inverse scattering transform for (1) involves solving a so-called DBAR problem [6]. A general study corresponding to all of the cases $\sigma^2 = \pm 1$, $\varepsilon^2 = \pm 1$ was undertaken in [7]. It was found that the solution to the initial value problem corresponding to the choice $\sigma^2 = \varepsilon^2 = 1$ requires the use of both a DBAR problem and non-local Riemann problem. The choices of σ^2 with $\varepsilon^2 = 1$ yield elliptic systems for which the initial value problem is no longer well posed; the fundamental problem is now to find solutions corresponding to data which give rise to well-posed problems. However, the issue of determining particular solutions was not addressed in sufficient detail in [7]. In this paper we focus on this particular matter, using the results of [7] to which we refer when necessary.

Special solutions which are expressible in terms of well known functions, such as solitons, are of both physical and mathematical interest. In this paper we describe how

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to obtain various classes of special solutions such as lump-type solutions, periodic configurations and line solitons.

One particular interesting subclass of solutions are those obtained when $\varepsilon^2 = -1$ for which $\theta(x, n, t)$ depends only on the radial coordinate $r = (x^2 + t^2)^{1/2}$ and n. In this case, $\theta(r, n)$ satisfies (we call this the radial Toda equation)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta_n}{\partial r}\right) = 2\sigma^2\{e^{2(\theta_{n+1}-\theta_n)} - e^{2(\theta_n - \theta_{n-1})}\}. \tag{1.2}$$

A method to obtain this class (as well as the more general case: $\theta(x, n, t)$) is described below.

For solutions of some finite-dimensional Toda systems see [10].

We begin by recalling the main facts we need.

2. Inverse problem

Equation (1.1) is the compatibility of the following linear problems

$$L\mu = (\partial_x + \varepsilon\theta_{n,t} - \theta_{n,x})\mu_n + i\sigma\{k\mu_{n+1} - (k + 1/k)\mu_n + C^2_{n-1}/k)\mu_{n-1}\} = 0 \tag{2.1}$$

$$M\mu = (\partial_t + 1/e(\theta_{n,x} - \theta_{n,t}))\mu_n + i\sigma/\varepsilon\{-k\mu_{n+1} + (k - 1/k)\mu_n + (C^2_{n-1}/k)\mu_{n-1}\} = 0 \tag{2.2}$$

where $C_n = e^{\theta_{n+1}-\theta_n}$, $k = k_R + ik_I$ is a spectral parameter while $\mu_n(x, t, k) = \mu(n, x, t, k)$ is a spectral 'wave' function. Equation (2.1) can be converted into an integral equation. Proper Green functions for this problem are (we take $\sigma = 1$; the case $\sigma^2 = -1$ is less interesting)

$$G_\pm(x, n, k) = \frac{1}{2\pi}\int_0^{2\pi}\{H(x)H(\pm g) - H(-x)H(\pm(-g))\, e^{-i(n\alpha + x\zeta)}\, d\alpha \tag{2.3}$$

where H is the Heaviside function, $k = R\, e^{i\varphi}$; $e^{i\alpha} = z$, $g(\alpha, \varphi) = \sin(\alpha - \varphi) + \sin(\varphi)$ and $\zeta = k/z + z/k - (k + 1/k)$. Functions G_+, G_- are defined for $R > 1$ and $R < 1$ respectively. Denote $\mu = \mu_+$ when $R > 1$, μ_- when $R < 1$. Then μ_+ and μ_- satisfy

$$\hat{R}_{k\pm\mu\pm}(x, n) = 1 \tag{2.4}$$

where operators $\hat{R}$ act as follows

$$\hat{R}_{k\pm}f(x, n) = f(x, n, k) - \int_{-\infty}^{\infty} dx \sum_{-\infty}^{\infty} G_\pm(x - x', n - n', k)(Vf)(x', n', k) \tag{2.5}$$

and

$$V\mu(x, n, k) = -\{\varepsilon\theta_{n,t} - \theta_{n,x})\mu_n + i(1/k)\{(C^2_{n-1} - 1)\mu_{n-1}\}. \tag{2.6}$$

The departure from holomorphicity of G is given by

$$\frac{\partial G_\pm(x, n, k)}{\partial \bar{k}} = \frac{\pm i\, \mathrm{sign}(k_R)}{2\pi \bar{k}}\left(\frac{-\bar{k}}{k}\right)^n e^{i(k + 1/k + (\bar{k} + 1/\bar{k})x}. \tag{2.6a}$$

Further

$$\frac{\partial G_\pm(x, n, k)}{\partial k} = \left\{\left(1 - \frac{1}{k_j^2}\right) ix - \frac{n}{k}\right\} G_\pm(x, n, k) \pm \frac{\mathrm{sign}(k_R)}{2\pi ik}. \tag{2.6b}$$

Equation (7) can be used to obtain the analytic properties of the eigenfunction μ. They are as follows: when $\sigma=1$ it possess both smooth regions of non-holomorphicity and a discontinuity across the unit circle which are measured by the 'scattering data' of the problem: $F(k)$, $k\in C$ and $D(z, k)$, where z and k are on the unit circle. We also allow for poles to exist at some points $k=k_j$ around which one has

$$\mu(x, n, k)=\nu(x, n, k)+\sum_j \frac{\phi_j(x, n)}{k-k_j} \tag{2.8}$$

where $\nu(x, n, k)$ stands for a non-singular function at $k=k_j$. Finally μ tends to 1 as $k\to\infty$. The equation of the inverse problem reflecting all of this analytic information is given by:

$$\begin{aligned}\mu(x, n, t, k)=1-\int_0^{2\pi} \mathrm{d}\alpha \int_0^{2\pi} &\mathrm{d}\psi \exp[\mathrm{i}\{(n+1)\psi-n\alpha+2x(\cos\alpha-\cos\psi)\}]\\ &\times\frac{\mu_-(x, n, t, \mathrm{e}^{\mathrm{i}\psi})D(\mathrm{e}^{\mathrm{i}\alpha}, \mathrm{e}^{\mathrm{i}\psi})}{\mathrm{e}^{\mathrm{i}\alpha}-k}\\ &-\frac{1}{4\pi^2}\int \frac{F(l, t)(-\bar{l}/l)^n \exp[i\{l+1/l+\bar{l}+1/\bar{l}\}]x}{\bar{l}(l-k)}\\ &\times\mu(x, n, t, -\bar{l})\,\mathrm{d}l_{\mathrm{R}}\,\mathrm{d}l_{\mathrm{I}}+\sum_j \frac{\phi_j(x, n)}{k-k_j}.\end{aligned} \tag{2.9}$$

The scattering data evolve in time as:

$$F(k, t)=F(k, 0)\exp[-\mathrm{i}t(k-1/k+\bar{k}-1/\bar{k}\}]/\varepsilon \tag{2.10a}$$

$$D(z, k, t)=D(z, k, 0)\exp[-\mathrm{i}t\{k-1/k+z-1/z\}]/\varepsilon. \tag{2.10b}$$

When $\sigma=\pm\mathrm{i}$ the non-local Rieman problem is no longer present; i.e. $D(z, k, 0)=0$. Since the former case is richer and more general than the latter we will analyse this, ie. in the sequel we take $\sigma=1$. Inserting appropriate scattering data into (2.4) one can obtain solutions $\mu(x, n, t, k)$ whereupon one has particular solutions to (2.1) by means of: $\theta(x, n, t)=\frac{1}{2}Ln\{\mu(x, n, t, k=0\}$. We refer the reader to [7] for proofs of the above facts. Below we consider special configurations in the spectral function.

3. Lump-type solutions

Lump solutions arise when we assume that $F(k)=D(z, k)=0$ and that μ has $2N$ poles off the unit circle at locations k_j and $-\bar{k}_j$ with residues at them ϕ_j, $-\bar{\phi}_j$ (this last requirement guarantees that $\theta(x, n, t)$ is real). Note that we are taking $\sigma=\varepsilon=1$. The equation of the inverse problem simply reflects these facts and reads

$$\mu(x, n)=1+\sum_j \frac{\phi_j(x, n)}{k-k_j}+\sum_j \frac{-\bar{\phi}_j(x, n)}{k+\bar{k}_j}. \tag{3.1}$$

To close this system one needs extra information relating the ϕ_j's with μ. The following important relationship applies

$$\lim_{k\to k_j}\left\{\mu(k)-\frac{\phi_j(x, n)}{k-k_j}\right\}=\left\{\left(1-\frac{1}{k_j^2}\right)\mathrm{i}x-\frac{n}{k_j}-\gamma_j\right\}\phi_j(x, n)+\lambda_j(x, n)x_j(x, n) \tag{3.2}$$

where γ_j and λ_j are arbitrary constants (discrete scattering data) and

$$\psi_j(x,n)=q_j\bar{\phi}_j$$
$$q_j=\left(\frac{-\bar{k}_j}{k_j}\right)^n \exp\left[\mathrm{i}\left(k_j+\frac{1}{k_j}+\left(\bar{k}_j+\frac{1}{\bar{k}_j}\right)\right]x. \tag{3.3}$$

We now prove equation (12). According to Fredholm theory ϕ_j solves (for definiteness sake we assume $|k_j|>1$)

$$\hat{R}_{k_j}\phi_j=0 \tag{3.4}$$

(since the k_j are off the unit circle the operator $\hat{R}_{k_j}$ is well defined). It is easily proven from (3.3) and (3.4) that ψ_j satisfies equation (3.4) too, i.e. there are two linearly independent homogeneous solutions at $k=k_j$. Using $\mu=\nu+\phi_j(x,n)/(k-k_j)$ in (2.4), where ν is regular at $k=k_j$, one has that $R(k)\nu=1-R(k)\phi_j(x,n)/(k-k_j)$. Taking the limit of this equation as k approaches k_j and using (7.2) it follows that

$$(R(k_j)\nu(x,n)=1+\beta_{1_j}+(R_{k_j}\rho_j)(x,n) \tag{3.5}$$

provided that

$$\beta_{2_j}=\int\sum_n V(k_j)\psi_j(x,n)\,\mathrm{d}x=0 \tag{3.6a}$$

and where we introduce

$$\beta_{1_j}=\frac{\operatorname{sign}(k_{R_j})}{2\pi\,\mathrm{i}k_j}\int\sum_n V(k_j)\phi_j(x,n)\,\mathrm{d}x \tag{3.6b}$$

$$\rho_j=\left\{\left(1-\frac{1}{k_j^2}\right)ix-\frac{n}{k_j}\right\}\phi_j. \tag{3.6c}$$

Define $h_j=\nu-\rho_j$. The (3.5) reads

$$(R(k_j)h)(x,n)=1+\beta_{1_j}. \tag{3.7}$$

Since the left-hand side is evaluated at $k=k_j$ for (3.7) to have a solution we need

$$\beta_{1_j}=-1. \tag{3.8}$$

Equation (3.8) follows from the well known Fredholm alternative. With this proviso the solution to (3.7) is a linear combination of linearly independent homogeneous solutions $\phi_j(x,n)$ and $\psi_j(x,n)$ at the point k_j. The claim follows.

We now take up the issue of temporal evolution of the discrete scattering data. Note that $M(k)\mu=0$. The representation (3.1) yields $M(k_j)\phi_j=0$. Letting k approach k_j one obtains that

$$\lim_{k\to k_j} M(k)\nu=-\mathrm{i}\lim_{k\to k_j}\{k-1/k-k_j+1/k_j\}\frac{\phi_j(x,n,t)}{k-k_j}$$
$$+\frac{C_{n-1}^2(x,t)}{k-k_j}\phi_j(x,n-1,t)\{1/k-1/k_j\}-\phi_j(x,n+1,t). \tag{3.9}$$

Using (3.4) and rearranging we finally obtain, after tedious calculations, that

$$\partial_t\gamma_j=\mathrm{i}(1+1/k_j^2) \tag{3.10a}$$

$$\partial_t\lambda_j=-\mathrm{i}(k_j-1/k_j+\bar{k}_j-1/\bar{k}_j)\lambda_j. \tag{3.10b}$$

Hence, the scattering data evolves in time as follows:

$$\gamma_j(t) = \gamma_j(0) + \mathrm{i}(1 + 1/k_j^2)t \tag{3.11a}$$

$$\lambda_j(t) = \lambda_j(0) \exp[-\mathrm{i}(k_j - 1/k_j + \bar{k}_j - 1/\bar{k}_j)t]. \tag{3.11b}$$

We now give formulae for the N-lump solution. Letting $k \to k_j$ and using (3.2) in equation (3.1) results in

$$1 + \sum_{j\neq i} \frac{\phi_j}{k_i - k_j} - \sum_{j=1}^{N} \frac{\bar{\phi}_j}{k_i + \bar{k}_j} + \frac{f_i \phi_i}{k_i} - \lambda_i q_i \bar{\phi}_i = 0 \tag{3.12}$$

where we introduce

$$f_i = \left(\frac{1}{k_i} - k_i\right) \mathrm{i}x + n + \gamma_i k_i.$$

Equation (3.12) along with its complex conjugate can be cast with $z_i = \phi_i / k_i$ as

$$1 + f_i z_i + \sum_{j\neq 1} \frac{k_j z_j}{k_i - k_j} - \sum_{j=1}^{N} \frac{\bar{k}_j \bar{z}_j}{k_i + \bar{k}_j} - \lambda_i q_i \bar{k}_i \bar{z}_i$$

$$= 1 + \bar{f}_i \bar{z}_i + \sum_{j\neq 1} \frac{\bar{k}_j \bar{z}_j}{\bar{k}_i - \bar{k}_j} - \sum_{j=1}^{N} \frac{k_j z_j}{\bar{k}_i + k_j} - \bar{\lambda}_i \bar{q}_i k_i z_i = 0. \tag{3.13}$$

We introduce the $2N \times 2N$ matrix

$$A_{\alpha,\beta} = (1 - \delta_{\alpha,\beta}) \frac{k_\beta}{k_\alpha - k_\beta} + \delta_{\alpha,\beta} f_\alpha - \delta_{\alpha+N,\beta} \lambda_\alpha q_\alpha$$

$\alpha, \beta = 1, \ldots, 2N$. Let $z_{j+N} = \bar{z}_j$, $k_{j+N} = -\bar{k}_j$, $\lambda_{j+N} = -\bar{\lambda}_j$, $\gamma_{j+N} = -\bar{\gamma}_j$. Then system (3.1) reads

$$\sum_\beta A_{\alpha,\beta} z_\beta = -1$$

and thus linear algebra yields that

$$\mu(x, n, k=0) = \frac{\operatorname{Det} \hat{A}(x, n)}{\operatorname{Det} A(x, n)} \tag{3.14}$$

where $\hat{A}_{\alpha,\beta}(x, n) = 1 + A_{\alpha,\beta}(x, n)$. It is easily proven that $\hat{A}_{\alpha,\beta}(x, n) = (k_\alpha / k_\beta) A_{\alpha,\beta}(x, n+1)$ whereupon manipulations in the corresponding determinant yields

$$\operatorname{Det} \hat{A}(x, n) = \operatorname{Det} A(x, n+1).$$

We refer to the resulting configuration as the N-lump solution; generically it depends upon $3N$ complex parameters l_j, γ_j, k_j. In general this solution is singular (note that in some physical contexts singular solutions could be more sensible than regular ones; e.g. equation (1.1) is related to Einstein's equations [8, 9] and one expects solutions to Einstein's equations to be singular somewhere). Nevertheless with an appropriate choice of parameters it is possible to obtain regular configurations. We show this for $N=1$. Thus let us assume μ to have two poles at $k_0 = R\,\mathrm{e}^{\mathrm{i}\varphi}$ and $-\bar{k}_0$ and let $\lambda \bar{k}_0 = \rho \mathrm{e}^{\mathrm{i}\Omega}$,

$k_0\gamma = w_R + i\omega_I$. The resulting configuration looks neater when viewed from a Lorentz frame (x', t') realted with (x, t) as follows

$$x' = \frac{x - vt}{(1-v^2)^{1/2}} \qquad t' = \frac{t - vz}{(1-v^2)^{1/2}} \qquad v = \frac{R^2-1}{R^2+1}.$$

The non-singular lump solution is given by

$$\theta(x,n,t) = \tfrac{1}{2}\,\mathrm{Ln}$$

$$1 + \frac{\{(1+2X+\rho\tan\varphi\sin(2n\varphi+4x'\cos\varphi+n\pi-\Omega)\}}{X^2+Y^2-\left\{\dfrac{1}{4\cos^2\varphi}+\rho^2+\rho\cos\varphi\cos(4x'\cos\varphi-(2n-1)\varphi+n\pi+\Omega)\right\}} \tag{3.15}$$

where $X = (n + \omega_R + 2x'\sin\varphi)$, $Y = 2t'\cos\varphi + \omega_I$; $\omega_I > \rho + 1/2\cos\varphi$ guarantees that the solution does not develop singularities upon time evolution.

The choice $\rho = 0$ results in the lump solution presented in [7]. Otherwise we obtain a more general configuration which shows both decaying behaviour in X, Y with oscillations.

Letting $\varphi = 0$ yields a x-independent solution, and hence a solution of the $1+1$-Toda lattice.

4. Solutions depending on arbitrary functions

We now consider solutions depending upon arbitrary functions. They correspond to the pure Riemann portion of the inverse problem. Therefore they only exist if $\sigma = 1$. To obtain these configuratons we solve the non-local Riemann problem (9) with $F(l, t) = 0$, $\phi_j = 0$. The ensuing equations can be explicitly solved provided the kernel of the problem is degenerate, i.e.

$$D(e^{i\alpha}, e^{i\psi}) = \sum_{i=1}^{N} A_i(e^{i\alpha}) B_i(e^{i\psi}).$$

With this ansatz equation (2.9) becomes

$$\mu(x,n,t,k) = 1 - \sum_{i=1}^{N} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\psi \exp[i\{(n+1)\psi - 2x\cos\psi) + 2(t/\varepsilon)\sin\psi\}\mu_-$$

$$\times (x,n,t,e^{i\psi}) B_i(e^{i\psi}) L_i(x,n,t,k) \tag{4.1}$$

where

$$L_i(x,n,t,k) = (-1/\pi)\int_0^{2\pi} \exp\{i(2x\cos\alpha - n\alpha) - 2(t/\varepsilon)\sin\alpha\}\frac{A_i(e^{i\alpha})}{e^{i\alpha}-k}\,d\alpha.$$

The integral equation (4.1) has a degenerate kernel and can therefore be solved in a straightforward way. Indeed we have with

$$F_i=\int_0^{2\pi}\exp[\mathrm{i}\{(n+1)\psi-2x\cos\psi)+2(t/\varepsilon)\sin\psi\}B_i(\mathrm{e}^{\mathrm{i}\psi})\,\mathrm{d}\psi \tag{4.2a}$$

$$E_{i,j}=\delta_{i,j}-\int_0^{2\pi}\exp[\mathrm{i}\{(n+1)\psi-2x\cos\psi+2(t/\varepsilon)\sin\psi\}]B_i(\mathrm{e}^{\mathrm{i}\psi})L_{j-}(x,n,t,\mathrm{e}^{\mathrm{i}\psi})\,\mathrm{d}\psi \tag{4.2b}$$

$$L_j(x,n,t,\mathrm{e}^{\mathrm{i}\psi})=\lim_{\substack{k\to 1^-\\ k=R\mathrm{e}^{\mathrm{i}\psi}}} L_j(x,n,t,k) \tag{4.2c}$$

that

$$\mu_-=1+\sum_{j=1}^{N} L_j(x,n,t,k)\xi_j \tag{4.3}$$

where the vector ξ_i solves the linear algebraic system

$$\sum_{j=1}^{N} E_{i,j}\xi_j=F_i, \qquad i=1,\ldots,N. \tag{4.4}$$

As long as Det $E\neq 0$ this system has a unique solution with the potential given by

$$\theta(x,n,t)=(\tfrac{1}{2})\ln\left\{1+\sum_{j=1}^{N}\mathrm{L_j(x,n,t,k=0)}\xi_j\right\}. \tag{4.5}$$

Thus for $N=1$ one has

$$\theta(x,n,t)=(\tfrac{1}{2})\,ln\left\{1+\frac{L(x,n,t,k=0)F(x,n,t)}{E(x,n,t)}\right\} \tag{4.6}$$

and it only remains evaluating the above quadratures for arbitrary A and B given. Although this is a straightforward problem it can become quite tedious even in the simplest cases. We present some examples:.

Here $\sigma=\varepsilon=1$.
Take

$$A=a\pi\{(\delta(a-\phi_0)+\delta(\alpha+\phi_0-\pi)\}$$

$$B=a\{\delta(\psi-\phi_1)+\delta(\psi+\phi_1-\pi)\}$$

where a, ϕ_0, ϕ_1 are just constants. Let also

$$l(x,n,t,\phi)=\begin{cases}2\cos((n+1)\phi-2x\cos(\phi)) & n \text{ even}\\ 2\sin((n+1)\phi-2x\cos(\phi)) & n \text{ odd.}\end{cases}$$

Then we obtain the following solution

$$\phi(x,n,t)=(\tfrac{1}{2})\ln\left\{1+\frac{a^2\mathrm{e}^{-r^2t}l(\phi_0)l(\phi_1)}{E(x,n,t)}\right\} \tag{4.7}$$

where

$$E(x,n,t)=1+a^2\,\mathrm{e}^{-r^2t}\left\{\frac{\sin\{l(\phi_0)-l(\phi_0)+(\phi_1-\phi_0)/2\}}{\sin\{(\phi_1-\phi_0)/2\}}\right.$$
$$\left.+(-1)^n\frac{\cos\{l(\phi_1)+l(\phi_0)+(\phi_1+\phi_0)/2\}}{\cos\{(\phi_1+\phi_0)/2\}}\right\} \tag{4.8}$$

and we define $\sin\phi_0-\sin\phi_1\equiv -r^2$. To have a non-singular solution take the parameter a to satisfy

$$|a|<\left\{\frac{1}{\sin\{(\phi_1-\phi_0)/2\}}+\frac{1}{\cos\{(\phi_1+\phi_0)/2\}}\right\}^{-1/2}.$$

Here $A=B=a/2\pi$.

We give a solution for the case corresponding to $\sigma=1$, $\varepsilon=-\mathrm{i}$ that can be expressed in terms of Bessel functions. It is a useful comment on the evaluation of the corresponding integrals. Consider, say, the integral F. Letting $z=\mathrm{e}^{\mathrm{i}\psi}$ we have

$$F=(a/2\pi)\int_{|z|=1} z^n\exp[-\mathrm{i}x(z+1/z)-t(z-1/z)]\,\mathrm{d}z.$$

We find it convenient to use

$$z^n\,\mathrm{e}^{-\mathrm{i}x(z+1/z)-t(z-1/z)}=\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{\{t-\mathrm{i}x\}^j\{-t-\mathrm{i}x\}^m z^{n+j-m}}{j!m!}$$

in the relevant integral. Noting that

$$\int_{|z|=1} z^n\,\mathrm{d}z=\delta_{n,-1}$$

one obtains that F is given by

$$F=a\left\{\frac{\mathrm{i}t-x}{\mathrm{i}t+x}\right\}^{(n+1)/2} J_{n+1}(2\sqrt{t^2+x^2}).$$

Second, one has

$$L(x,n,t,k)=\int_{|z|=1}\frac{\exp[\mathrm{i}x(z+1/z)+t(z-1/z)]\,dz}{z^{n+1}(z-k)}.$$

The integrand has two poles. The one at $z=k$ contributes as

$$\frac{\exp[\mathrm{i}x(k+1/k)+t(k-1/k)}{k^{n+1}}.$$

To obtain the contribution at $z=0$ we use an expansion in powers of (k/z)

$$\frac{\exp[\mathrm{i}x(z+1/z)+t(z-1/z)}{z^{n+2}(1-k/z)}=\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\{\mathrm{i}x-t\}^j\{t+\mathrm{i}x\}^m k^l z^{j-m-n-l-2}}{j!m!}$$

which turns out to be convergent for $|k/z|<1$ and hence for $|z|=1$ (recall that $k=1^-$ in the 'L' integral). This enables one to commute the series with integrals. With this in mind the relevant calculations do not offer any further problem and go in much the same way as above; hence we gloss over the details. The solution reads

$$\theta(x,n,t)=(\tfrac{1}{2})\ln\left\{1+\frac{a^2J_{n+1}^2(2\sqrt{t^2+x^2})}{1-a^2\left(1+\sum_{j=n+1}^{\infty}J_j^2(2\sqrt{t^2+x^2})\right)}\right\} \tag{4.9}$$

where J_n stands for the Bessel function of order n and we assume $a^2<\frac{1}{2}$. Note that the solution is nowhere singular and decays as $x^2+n^2+t^2\to\infty$ as predicted in [7] via asymptotic analysis for the solutions of (1.1) with $\sigma=1$, $\varepsilon=-\mathrm{i}$. Equation (4.9) satisfies the 'radial' Toda equation (1.2) and plays the role of a soliton for this equation.

More general radial 'N-soliton' solutions can be obtained simply taking

$$A_i=B_i=a_i/2\pi \qquad i=1,\ldots,N$$

and then solving the corresponding linear system (4.4); note that all the necessary quadratures would not differ from those already evaluated.

5. Periodic solutions

There also exists an interesting class of periodic solutions obtainable from the above picture. Spectrally they correspond again to the Riemann-problem portion of the spectrum and hence they only exist for $\sigma=1$. In (2.9) take $\sigma=\varepsilon=1$, $F(l,0)=\phi_j=0$ and

$$D(\mathrm{e}^{\mathrm{i}\alpha},\mathrm{e}^{\mathrm{i}\psi})=-\sum_{i=1}^{N}\{r_i\delta(\psi-\psi_i)\delta(\alpha-\hat{\alpha}_i)-\bar{r}_i\delta(\psi-\psi_i')\delta(\alpha-\hat{\alpha}_i')\} \tag{5.1}$$

where $\psi_i\varepsilon(\pi/6, 5\pi/6)$ and $\hat{\psi}_i$ is obtained from ψ_i by requiring that

$$2\cos(\psi_i)+\chi_i=2\cos(\hat{\chi}_i)+\hat{\chi}_i$$

and $\hat{\psi}_i\neq\psi_i$. The r's are discrete scattering data which evolve in time via (2.10b) which reads

$$r_i(t)=r_i(0)\exp\{2t(\sin\hat{\psi}_0-\sin\psi_0)\}. \tag{5.2}$$

Note that this case cannot be incorporated into the general framework of the last section; indeed the kernel D is longer factorizable. Insertting the above ansatz into equation (2.9) one obtains at once a linear algebraic system which can be solved to give particular real configurations $\theta(x,n,t)$. The simplest case corresponds to taking $N=1$ in (5.1). Equation (2.9) reads with $p=\psi_1-\hat{\psi}_1$ and $x'=x+n$ as

$$\mu(k)=1+\frac{r\mathrm{e}^{\mathrm{i}px'}}{\mathrm{e}^{\mathrm{i}\hat{\psi}_1}+k}\mu(\mathrm{e}_1^{\mathrm{i}\psi})+\frac{\bar{r}\,\mathrm{e}^{-\mathrm{i}px'}}{\mathrm{e}_1^{-\mathrm{i}\hat{\psi}}+k}\mu(-\mathrm{e}^{-\mathrm{i}\psi_1})$$

which results in the system

$$\mu(\mathrm{e}^{\mathrm{i}\psi_1})=1+\frac{r\mathrm{e}^{\mathrm{i}px'}}{\mathrm{e}^{\mathrm{i}\hat{\psi}_1}+\mathrm{e}^{\mathrm{i}\psi_1}}\mu(\mathrm{e}_1^{\mathrm{i}\psi})+\frac{\bar{r}\,\mathrm{e}^{-\mathrm{i}px'}}{\mathrm{e}^{-\mathrm{i}\hat{\psi}_1}+\mathrm{e}^{\mathrm{i}\psi_1}}\mu(-\mathrm{e}^{-\mathrm{i}\psi_1}) \tag{5.3a}$$

$$\mu(-\mathrm{e}^{-\mathrm{i}\hat{\psi}_1})=1+\frac{r\mathrm{e}^{\mathrm{i}px'}}{\mathrm{e}^{\mathrm{i}\hat{\psi}_1}+\mathrm{e}^{-\mathrm{i}\hat{\psi}_1}}\mu(\mathrm{e}^{\mathrm{i}\psi_1})+\frac{\bar{r}\mathrm{e}^{-\mathrm{i}px'}}{\mathrm{e}^{-\mathrm{i}\hat{\psi}_1}-\mathrm{e}^{-\mathrm{i}\hat{\psi}_1}}\mu(-\mathrm{e}^{-\mathrm{i}\psi_1}). \tag{5.3b}$$

Upon solving this system one obtains a solution which is bounded and $2\pi/p$ periodic in x'. Ast $t \to \infty$ the solution decays exponentially in time. The explicit form of this configuration is

$$\theta(x, n, t) = (\tfrac{1}{2}) \ln\{h(x'+1, t)/h(x', t)\} \tag{5.4}$$

where

$$h(x', t) = 1 + \frac{e^{2\omega t}}{\rho^2} \frac{\sin^2(p/2)}{\cos^2(l/2)} - 2(e^{\omega t}/\rho)\cos(px + x_0) \tag{5.5}$$

and

$$\frac{r}{e^{i\bar{\psi}_1} - e^{i\psi_1}} = \rho e^{ix_0} \qquad l = \bar{\psi}_1 + \psi_1 \qquad \omega = (16\sin^2(p/2) - p^2)^{1/2}$$

and for boundedness we take $\rho < \frac{1}{2}$.

6. Line solitons and instantons

When $\sigma^2 = \varepsilon^2 = -1$ equation (1.2) admits a 'separation of variables' solution. It follows assuming that both sides of equation (1.2) equal $nf(r)$. Equating first the right side this ansatz results in a linear equation for θ_n which is trivially solved as

$$\theta_{n+1} - \theta_n = \tfrac{1}{2} \ln\left\{A(r) + B(r)n + \frac{f(r)}{2} n^2\right\}$$

where A, B are arbitrary functions of r. To simplify matters assume $A(r) = af(r)/2$, $B(r) = bf(r)/2$. Taking this result into the left-hand side of equation (1.2) yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f(r)}{\partial r}\right) = -4f(r).$$

Upon integration we obtain that

$$f(r) = \frac{2c}{(r^2 + c)^2}$$

(c is a constant) and hence that

$$\theta_{n+1} - \theta_n = \tfrac{1}{2} \ln\left\{2c\frac{a + bn + n^2}{(r^2 + c)^2}\right\} \tag{6.1}$$

$c > 0$, $4a - b^2 > 0$ guarantees that the resulting configuration is nowhere singular. This solution cannot be obtained from the IST analysis of reference [7] since it does not fall into the class of 'potentials' considered there; nevertheless it is a very interesting solution. Indeed setting $x' = \varepsilon x$, $t' = \varepsilon t$, $z = \varepsilon n$, and letting $\varepsilon \to 0$ yields, after trivial rescaling of the constants, that

$$\theta_{n+1} - \theta_n \to \omega(x', t', z) = \tfrac{1}{2} \ln\left\{2\frac{a' + b'z + 2c'z^2}{(r^2 + c')^2}\right\} \tag{6.2}$$

which is nothing but the 'Eguchi–Hanson' gravitational instanton [8, 9]; it solves the long wave limit Toda field equation

$$(\partial_{x'x'} + \partial_{t't'})\omega(x', t', z) = \partial_{zz}\, e^{\omega(x', t', z)}. \tag{6.3}$$

Finally, we mention that equation (1.1) also admits a line-soliton configuration for all choices of the parameters ε and σ; we skip here their spectral interpretation and simply write it out. If $\varepsilon = 1$, $\sigma = \mathrm{i}$ the solution reads

$$\theta(x, n, t) = (\tfrac{1}{2}) \ln \left\{ 1 + (e^{p} - 1) \frac{e^{x_0 + p(x+n) + \omega t}}{1 + e^{x_0 + p(x+n) + \omega t}} \right\} \tag{6.4}$$

where p is a parameter and $\omega = \sqrt{p^2 + 4(e^{p/2} - e^{-p/2})^2}$.

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